Phase Space Tomography

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UMER Course, USPAS 2008
Outline

1. Phase Space Definition
2. Transport Matrix Definition
3. History/Overview of Tomography
4. Tomography for Beams with Space Charge
5. Overview of Lab Exercise
Introduction to Beam Phase Space

- Beams are not perfect laminar!

- Information of the transverse velocity distribution is needed to quantify the quality of the beam
Introduction to Beam Phase Space

- Paraxial limit: \[ p \approx p_z \]
  \[ X' = \frac{dx}{dz} = \frac{p_x}{p_z} \approx \frac{p_x}{p} \]

- Phase space:

- Beam Emittance: useful measure of how far away the beam is from laminar
Example of Importance of Phase Space

- Initial distribution
- Downstream
Beam Transport Matrix

\[
\begin{pmatrix}
    r_0 \\
r'_0
\end{pmatrix}
= \begin{pmatrix}
m_{11} & m_{12} \\
m_{21} & m_{22}
\end{pmatrix}
\begin{pmatrix}
r_0 \\
r'_0
\end{pmatrix}
= \mathbf{M}
\begin{pmatrix}
r_0 \\
r'_0
\end{pmatrix}
\]

Transfer matrix of the element

Assumption: No Space Charge

\[
\prod_{i} M_i = \text{........} M_3 M_2 M_1
\]
Introduction to Tomography

- An object in n-dimensional space can be recovered from a sufficient number of projections onto (n-1)-dimensional space
Phase-Space Tomography

Question: How we can rotate the phase-space distribution?

- How we can get projections of Phase Space?
- How to account for Phase Space Stretching?
Phase Space Projections

Screen Image

Phase Space (at screen)

\[ c(x) = \int \int \int f(x, x', y, y') dx \, dy \, dy' = \mu_\theta(x) \]

What is \( \theta \)?
Angles of Projections and Scaling Factors

\[
\begin{pmatrix}
x_1 \\
x_1'
\end{pmatrix} =
\begin{pmatrix}
m_{11} & m_{12} \\
m_{21} & m_{22}
\end{pmatrix}
\begin{pmatrix}
x_0 \\
x_0'
\end{pmatrix} =
\begin{pmatrix}
x \\
x'
\end{pmatrix} =
\begin{pmatrix}
s_1 & 0 \\
\frac{\sqrt{s_1^2 s_2^2 - 1}}{s_1} & \frac{1}{s_1}
\end{pmatrix}
\begin{pmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{pmatrix}
\begin{pmatrix}
x_0 \\
x_0'
\end{pmatrix}
\]

\[s_1 = \sqrt{m_{11}^2 + m_{12}^2}\]
\[s_2 = \sqrt{m_{21}^2 + m_{22}^2}\]

\[\theta = \tan^{-1}\left(\frac{m_{12}}{m_{11}}\right)\]

M: Transport Matrix

Transform Matrix

Rotation Matrix

McKee et al. NIMA (1995)
Stratakis et al., PRST-AB (2006)
Beams with Space-Charge (SC)

- **Motion:** \( \frac{d^2 x}{dz^2} = -\kappa_0 x + F_{SC} \)
- **SC Force:** \( F_{SC} = F_l + F_{nl} = \frac{K}{R(z)^2} x + F_{nl} \)
- **Linear Approximation:**
  \[
  \frac{d^2 x}{dz^2} = (-\kappa_0 + \frac{K}{R(z)^2})x = \kappa x
  \]
- **Problem:** For beams with SC beams \( \kappa \) depends on \( R \).
Beam radius is determined by:

\[ R'' + \kappa R \frac{K}{R} - \frac{\varepsilon^2}{R^3} = 0 \]

\[ K = \frac{qI}{2\pi\varepsilon_0 mv^3} \]

- a: rms beam radius
- \( K \): beam perveance
- \( \varepsilon \): rms beam emittance
- \( \kappa \): lens focusing function
Overview of Tomography Experiment

• **GOAL:** Measure the beam phase space and beam emittance by using Tomography
Overview of Tomography Experiment

- **BEAM DISTRIBUTIONS:**
  - One more “uniform beam”
  - One highly nonuniform
Experiment Configuration

- **Transport Line:**
  - 1 Solenoid
  - 2 drift Sections

\[ M = M_{D2}M_S M_{D1} \]
Transport Matrix Calculation

- Run Code: `ScalF_RotA_Calc.m`
- Code is solving the envelope along the transport line
  \[ R^* + \kappa R - \frac{K}{R} - \frac{\varepsilon^2}{R^3} = 0 \]
- Divide transport line at many hard edge elements and get the transport matrix for each step

\[
\kappa = (-\kappa_0 + \frac{K}{R(z)^2})
\]

- Net Transport Matrix: \[ M = M_{i+2}M_{i+1}M_i \ldots \]
Beam Photo Collection

- Use the given currents and vary the strength of solenoid and save the beam photos on the screen.

- Note that photos are not centered.
Centering Photos and Image Analysis

- Run code PhotoProcessing.m

- Each image can be thought as a 2D matrix, $G$, where the elements $G(i,j)$ represent the intensity values at a given pixel.

- Beam Centroid: $x_c = \sum_i \sum_j i G(j,i) / I$  
  $y_c = \sum_i \sum_j j G(j,i) / I$

- Photo Centering: $x_c \rightarrow x_c - (x_c - N/2)$  
  $y_c \rightarrow y_c - (y_c - N/2)$
Beam Tomography

- Run Tomography.m
Phase Space Reconstruction
Beam Emittance

• Run **EmitCalc.m**. This code gives you the beam emittance but **WILL NOT** given to you. Its your homework!

• Beam Emittance $\epsilon_x = 4\sqrt{\langle x^2 \rangle < x'^2 > - < xx' >^2}$

\[
\langle x^2 \rangle = T^2 \left[ \sum_{i=1}^{N} \sum_{j=1}^{N} i^2 M(j, i) \right] / I
\]

\[
\langle x'^2 \rangle = T^2 \left[ \sum_{i=1}^{N} \sum_{j=1}^{N} j^2 M(j, i) \right] / I
\]

\[
\langle xx' \rangle = T^2 \left[ \sum_{i=1}^{N} \sum_{j=1}^{N} (ij) M(j, i) \right] / I
\]

\[
\epsilon_x = \frac{4T^2}{I} \sqrt{\left[ \sum_{i=1}^{N} \sum_{j=1}^{N} i^2 M(j, i) \right] \left[ \sum_{i=1}^{N} \sum_{j=1}^{N} j^2 M(j, i) \right] - \left[ \sum_{i=1}^{N} \sum_{j=1}^{N} (ij) M(j, i) \right]^2}
\]

T: mm/pixel
• Good Luck!
Tomography Example

Transport Matrix: \[ M = M_D M_Q \]

Quadrupole Matrix: \[ M_Q = \begin{pmatrix} \cos(\sqrt{\kappa L_1}) & \frac{1}{\sqrt{\kappa}} \sin(\sqrt{\kappa L_1}) \\ -\sqrt{\kappa} \sin(\sqrt{\kappa L_1}) & \cos(\sqrt{\kappa L_1}) \end{pmatrix} \]

Drift Matrix: \[ M_D = \begin{pmatrix} 1 & L_2 \\ 0 & 1 \end{pmatrix} \]
Tomography Example

Transport Matrix:

\[
M_1 = \begin{pmatrix}
-L_2 \sqrt{\kappa_0} \sin(\sqrt{\kappa_0} L_1) + \cos(\sqrt{\kappa_0} L_1) & \frac{1}{\sqrt{\kappa_0}} \sin(\sqrt{\kappa_0} L_1) + L_2 \cos(\sqrt{\kappa_0} L_1) \\
-\sqrt{\kappa_0} \sin(\sqrt{\kappa_0} L_1) & \cos(\sqrt{\kappa_0} L_1)
\end{pmatrix}
\]

Rotation Angle:

\[
\theta = \tan^{-1}\left(\frac{m_{12}}{m_{11}}\right) = \tan^{-1}\left(\frac{\frac{1}{\sqrt{\kappa_0}} \sin(\sqrt{\kappa_0} L_1) + L_2 \cos(\sqrt{\kappa_0} L_1)}{-L_2 \sqrt{\kappa_0} \sin(\sqrt{\kappa_0} L_1) + \cos(\sqrt{\kappa_0} L_1)}\right)
\]

Scaling Factor:

\[
s = \sqrt{m_{11}^2 + m_{12}^2}
\]

\[
s = \sqrt{\left[\frac{1}{\sqrt{\kappa_0}} \sin(\sqrt{\kappa_0} L_1) + L_2 \cos(\sqrt{\kappa_0} L_1)\right]^2 + \left[-L_2 \sqrt{\kappa_0} \sin(\sqrt{\kappa_0} L_1) + \cos(\sqrt{\kappa_0} L_1)\right]^2}
\]
Tomography Example

\[ g(x,y) \]

\[
M_1 = \begin{pmatrix}
-0.70 & 0.12 \\
-12.00 & 0.76 \\
\end{pmatrix}
\]

\[
M_1 = \begin{pmatrix}
0.40 & 0.14 \\
-4.20 & 0.92 \\
\end{pmatrix}
\]

\[
M_1 = \begin{pmatrix}
1.49 & 0.17 \\
3.48 & 1.06 \\
\end{pmatrix}
\]

\[ c(x) \]

\[ s = 0.71 \quad \theta = 169.7^\circ \]

\[ s = 0.43 \quad \theta = 20.2^\circ \]

\[ s = 1.50 \quad \theta = 6.4^\circ \]